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Hydrodynamic Instability in a Single Diffusive Bottom Heavy System with Thermally Insulating Permeable Boundaries

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Abstract

We utilize the modified linear stability analysis to establish mathematically, the existence of hydrodynamic instability in a single diffusive bottom heavy system under the force field of gravity, for situations with more general nature of the bounding surfaces, in the parameter regime $\alpha_2 T_0 > 1$, where T_0 and α_2 being some properly chosen mean temperature and coefficient of variation of specific heat at constant volume due to temperature variation of the fluid, respectively.

Keywords: Buoyancy, Convection, Gravity, Insulating, Linear stability, Permeable.

1. Introduction

Banerjee and Gupta [1] have given the modified analysis of the Rayleigh's [2] classical linear stability theory of thermal instability problem, by taking into account of the fact that the linear theoretical explanation of buoyancy driven thermal instability in a horizontal layer of fluid heated from below or above, also known as the modified Bénard [3, 4] instability problem, should depend not only upon the Rayleigh number which is proportional to the uniform temperature difference maintained across the layer but also upon another parameter that arises due to the variation in the specific heat at constant volume on account of the variations in temperature. Recently, Gupta and Shandil [5] established the existence of instability in a single diffusive bottom heavy system for thermally conducting permeable boundaries using the modified analysis.

In this paper, we prove the existence of hydrodynamic instability in a single diffusive bottom heavy system with thermally insulating permeable boundaries. The thermally insulating boundary conditions have several physical justifications that arise from a more accurate description of heat transfer phenomenon in the environment surrounding the fluid. The problem under investigation helps in better understanding of thermal convection, apart from its importance and applications in many scientific and engineering fields. The characteristic value problem is solved by using the Galerkin technique. Further, it is observed that the limiting cases of the boundary parameters K_0 an K_1 characterizing the permeable nature of the lower and upper boundary respectively, give rise to the particular cases, namely, when both the bounding surfaces are either dynamically free $(K_0 \to 0, K_1 \to 0)$ or both rigid $(K_0 \to \infty, K_1 \to \infty)$, and either one of them is dynamically free $(K_0 \to 0$ or $K_1 \to 0)$ while the other one is rigid $(K_0 \to \infty$ or $K_1 \to \infty)$.

2. Formulation of the Eigenvalue Problem in Non-Dimensional Form

A viscous finitely heat conducting modified Boussinesq liquid layer of infinite horizontal extension and finite vertical depth is statically confined between two horizontal boundaries at $z = 0$ and $z = d$ which are respectively maintained at uniform temperatures T_0 and T_1. We choose Cartesian coordinate system with the x and y axes in the plane of the lower boundary and the positive direction of the z-axis along the vertically upward direction. Further, both the bounding surfaces are thermally insulating and permeable. We mathematically analyze the onset of hydrodynamic instability of the system under the force field of gravity. The non-dimensional form of the modified governing linearized perturbation equations, which govern the initiation of thermal convection, are given by Banerjee and Gupta [1] as

$$(D^2 - a^2)\left(D^2 - a^2 - \frac{p}{P_r}\right)w = \theta, \tag{2.1}$$

$$(D^2 - a^2 - p)\theta = -Ra^2(1-\alpha_2 T_0)w, \tag{2.2}$$

where w is the z-component of the perturbation velocity, θ is the temperature perturbation, a is the horizontal wave number, $P_r = \nu / \kappa$ is the thermal Prandtl number, $R = g\alpha\beta d^4 / \kappa\nu$ is the Rayleigh number, α is the volume coefficient of thermal expansion, $\beta = (T_0 - T_1)/d$ is the maintained temperature gradient, g is the gravitational acceleration, ν is the kinematic viscosity, κ is the thermal diffusivity, $p = p_r + ip_i$ represents the growth rate of perturbations (a complex constant in general), p_r and p_i being real constants, and $D = d/dz$.

Since both the lower and upper boundary planes are fixed and thermally insulating, the associated boundary conditions are:

$$w = 0 \quad D\theta = 0 \text{ at } z = 0 \text{ and } z = 1. \tag{2.3}$$

Further, Beavers and Joseph [6] proposed that at a permeable boundary the normal derivative of the tangential velocity is directly proportional to that velocity and if the normal is taken into the fluid then the constant of proportionality is positive. As described by Gupta et al. [7], the appropriate boundary conditions are given by

$$D^2 w - K_0 Dw = 0, \text{ at } z = 0, \tag{2.4}$$

$$D^2 w + K_1 Dw = 0, \text{ at } z = 1, \tag{2.5}$$

where K_0 and K_1 are non-negative dimensionless parameters, characterizing the permeable nature of the lower and upper boundary respectively.

Eqns. (2.1)-(2.2) together with boundary conditions (2.3)-(2.5) pose a double eigenvalue problem for p, for prescribed values of a, P_r, R, $\alpha_2 T_0$, K_0 and K_1. The given normal mode is stable, neutral or unstable according as the real part p_r of p is negative, zero or positive respectively. Further, the marginal state of the system is defined by $p_r = 0$, and if $p_r = 0$ implies that $p_i = 0$ for every wave number a then the ensuing thermal convection is neutral and the 'principle of exchange of stability' is valid. Otherwise, we will have over-stability at least when instability sets in as a certain mode.

3. The Marginal State and Solution of the Problem

Case I: When Liquid Layer is Heated From Below (i.e. *R* > 0)

(A) When $1-\alpha_2 T_0 > 0$.

The governing Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) imply that we have the modified Bénard thermal instability problem with insulating permeable boundaries, wherein the liquid layer is heated from below, in the parameter regime $\alpha_2 T_0 < 1$. The Pellew and Southwell [7] technique for the characterization of the marginal state is applicable with the following result.

Theorem 1. If $1-\alpha_2 T_0 > 0$ with $R > 0$, a necessary condition for the existence of nontrivial solutions for w and θ satisfying Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) is that $p_i = 0$. (3.1)

Proof. Multiplying Eq. (2.1) throughout by w^* (the complex conjugate of w), and integrating the resulting equation over the vertical range of z, and substituting for w^* in this equation from Eq. (2.2), we then integrate each term of the equation so obtained, by parts, for a suitable number of times with the help of boundary conditions (2.3)-(2.5) and derive from the imaginary part of the integrated equation

$$p_i\left[\frac{1}{P_r}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{1}{R(1-\alpha_2 T_0)a^2}\int_0^1|\theta|^2\,dz\right] = 0 . \qquad (3.2)$$

From Eq. (3.2), it follows that $p_i = 0$.

This implies that the 'principle of exchange of stabilities' is valid for the problem under consideration and hence the marginal state is characterized by $p = 0$. In this case, Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) can be treated as an eigenvalue problem in R for given values of a, $\alpha_2 T_0$, K_0 and K_1. We proceed along the same lines as described by Gupta and Kalta [8], using the Galerkin technique to obtain the following results.

Theorem 2. If $1-\alpha_2 T_0 > 0$ with $R > 0$ and $p = 0$, a nontrivial solution for w and θ satisfying Eqn. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) implies that the Rayleigh number R in terms of a, $\alpha_2 T_0$, K_0 and K_1 is given by

$$\begin{aligned} R = {} & \frac{1}{(1-\alpha_2 T_0)} \times \frac{10}{7\{K_0(K_1+9)+9(K_1+8)\}^2} \times \\ & \big[504\{K_0(K_1+4)+4(K_1+3)\}\{K_0(K_1+9)+9(K_1+8)\} \\ & +24a^2[72\{K_1(K_1+13)+51\}+3K_0\{5K_1(K_1+14)+312\}+K_0{}^2\{K_1(K_1+15)+72\}] \\ & +a^4[76K_1(K_1+15)+K_0\{17K_1(K_1+16)+1140\}+K_0{}^2\{K_1(K_1+17)+76\}+4464]\big]. \end{aligned} \qquad (3.3)$$

For given values of $\alpha_2 T_0$, K_0 and K_1, equation (3.3) gives the Rayleigh number R as a function of the wave number a . The minimum of R is the critical Rayleigh number R_c and the value of a at which R attains minimum is the critical wave number a_c . A close observation of the expression for R given by equation (3.3) shows that R attains its minimum when $a = 0$. We put $a = 0$ on the right hand side of the equation (3.3) and obtain R_c as

$$R_c = \frac{720}{(1-\alpha_2 T_0)}\left[\frac{K_0 K_1 + 4(K_0+K_1)+12}{K_0 K_1 + 9(K_0+K_1)+72}\right]. \qquad (3.4)$$

Remark 1. It is easily seen from Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) that for the case when $K_0 \to 0$ and $K_1 \to 0$, we have governing equations for the modified Bénard problem with both boundaries free and from equation (3.4) we find that

$$R_c = \frac{120}{(1-\alpha_2 T_0)}. \tag{3.5}$$

When $\alpha_2 T_0 = 0$ the value R_c = 120 is exactly the same as that obtained by Nield [9] corresponding to this case.

Remark 2. When $K_0 \to \infty$ and $K_1 \to \infty$, it is easily seen from Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) that we have governing equations for the modified Bénard problem with both boundaries rigid and in this case we find from Eq. (3.4) that

$$R_c = \frac{720}{(1-\alpha_2 T_0)}. \tag{3.6}$$

When $\alpha_2 T_0 = 0$ the value R_c = 720 is exactly the same as that obtained by Sparrow et al. [10] corresponding to this case.

Remark 3. When either ($K_0 \to 0$ and $K_1 \to \infty$) or ($K_0 \to \infty$ and $K_1 \to 0$), it is easily seen from Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) that we have governing equations for the modified Bénard problem when either one of them is dynamically free while the other is rigid and in this case we find from equation (3.4) that

$$R_c = \frac{320}{(1-\alpha_2 T_0)}. \tag{3.7}$$

This expression (3.7) is identical with that obtained by Gupta and Surya [11] corresponding to this case.

Since R is positive the initial distribution of density is top heavy and therefore potentially gravitationally unstable. This destabilizing effect together with the joint stabilizing effects of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1-\alpha_2 T_0 > 0$. Above described theorems 1 and 2 imply that for the modified Bénard problem with insulating permeable boundaries, in which the liquid layer is heated from below, instability must set in the system when R goes beyond a critical value with the 'principle of exchange of stabilities' being valid at the marginal state.

(B) When $1-\alpha_2 T_0 < 0$.

Here it is pointed out that nature of the problem is altogether different in the regime $1-\alpha_2 T_0 < 0$ with $R > 0$, in which case we have $R(1 - \alpha_2 T_0) < 0$, and this in turn forces all the perturbations to decay, thus making the system stable.

We now prove the following theorem to show the existence of this new stabilizing mechanism.

Theorem 3. If $1-\alpha_2 T_0 < 0$ with $R > 0$, a necessary condition for the existence of nontrivial solutions for w and θ satisfying Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) is that

$$p_r < 0. \tag{3.8}$$

Proof. Multiplying Eq. (2.1) throughout by w^* (the complex conjugate of w), and integrating the resulting equation over the vertical range of z, and substituting for w^*

in this equation from equation (2.2), we then integrate each term of the equation so obtained by parts for a suitable number of times with the help of boundary conditions (2.3)-(2.5) and derive from the real part of the integrated equation that

$$K_1\left(|Dw|^2\right)_1 + K_0\left(|Dw|^2\right)_0 + \int_0^1\left(\left|D^2w\right|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + p_r\left[\frac{1}{P_r}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz\right]$$
$$= \frac{1}{R|1-\alpha_2T_0|a^2}\int_0^1\left(|D\theta|^2 + a^2|\theta|^2 + p_r|\theta|^2\right)dz \tag{3.9}$$

From equation (3.9), it follows that $p_r < 0$.

Case II: When Liquid Layer is Heated From Above (i.e. $R < 0$)

(A) When $1-\alpha_2T_0 > 0$.

The governing Eqns. (2.1)-(2.2) and boundary conditions (2.3)-(2.5) imply that we have the modified Bénard thermal instability problem with insulating permeable boundaries, wherein the liquid layer is heated from above, in the parameter regime $1-\alpha_2T_0 > 0$. Further, the stability of the system can be established along the classical lines as given in Chandrasekhar [12] so that any oscillation which may exist in the system must of necessity decay. We have the following theorem:

Theorem 4. If $1-\alpha_2T_0 > 0$ with $R < 0$, a necessary condition for the existence of nontrivial solutions for w and θ satisfying Eqns. (2.1)-(2.2), and (2.3)-(2.5) is that $p_r < 0$. (3.10)

Proof. Proceeding exactly as in the proof of Theorem 3 we have in place of Eq. (3.9)

$$K_1\left(|Dw|^2\right)_1 + K_0\left(|Dw|^2\right)_0 + \int_0^1\left(\left|D^2w\right|^2 + 2a^2|Dw|^2 + a^4|w|^2\right)dz + p_r\left[\frac{1}{P_r}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz\right]$$
$$= \frac{1}{|R|(1-\alpha_2T_0)a^2}\int_0^1\left(|D\theta|^2 + a^2|\theta|^2 + p_r|\theta|^2\right)dz \tag{3.11}$$

where $\left(|Dw|^2\right)_0$ and $\left(|Dw|^2\right)_1$ are values of $|Dw|^2$ at the lower and upper boundary surface respectively. From equation (3.11), it follows that $p_r < 0$.

Theorem 4 establishes the stability of the system when the liquid layer under consideration is heated from above. It may, however, be remarked that the usual physical circumstances are characterized by parameter regime $1-\alpha_2T_0 > 0$ and it is only in this parameter regime that the above results are valid.

(B) When $1-\alpha_2T_0 < 0$.

The governing equations and boundary conditions imply that we have the modified Bénard problem with indulating permeable boundaries, wherein liquid layer is heated from above, in the parameter regime $1-\alpha_2T_0 < 0$. Since R is negative the initial distribution of density is bottom heavy and therefore statically gravitationally stable. This stabilizing effect together with the joint stabilizing effect of viscosity and conduction is expected to impart, in the usual parameter regime characterized by $1-\alpha_2T_0 > 0$ an overall stabilizing effect to the system. That this is really the case is borne out by Theorem 4 wherein the stability of the system is proved in such situations. The nature of the problem, however, is completely different in the regime $1-\alpha_2 T_0 < 0$ with R < 0, in which case we have $R(1 - \alpha_2 T_0) > 0$. This in turn introduces a new instability into the system as $|R|$ goes beyond a critical value with the 'principle of exchange of stabilities' being valid at the marginal state.

We now prove the following two theorems to show the existence of this new instability mechanism.

Theorem 5. If $1-\alpha_2 T_0 < 0$ with $R < 0$, a necessary condition for the existence of nontrivial solutions for w and θ satisfying Eqns. (2.1)-(2.2), and (2.3)-(2.5) is that

$$p_i = 0. \tag{3.11}$$

Proof. Proceeding exactly as in the proof of Theorem 1 we have in place of Eq. (3.2)

$$p_i\left[\frac{1}{P_r}\int_0^1\left(|Dw|^2 + a^2|w|^2\right)dz + \frac{1}{|R(1-\alpha_2 T_0)|a^2}\int_0^1|\theta|^2\,dz\right] = 0 \tag{3.13}$$

From equation (3.13), it follows that $p_i = 0$. It implies that the 'principle of exchange of stabilities' is valid in the present case and hence the marginal state is characterized by $p = 0$. In this case, Eqns. (2.1)-(2.2) and (2.3)-(2.5) can be treated as an eigenvalue problem in R for given values of a^2, $\alpha_2 T_0$, K_0 and K_1. Proceeding exactly along the same lines as described by Gupta and Kalta [8], using the Galerkin technique to obtain the following results.

Theorem 6. If $1-\alpha_2 T_0 < 0$ with $R < 0$ and $p = 0$, a nontrivial solution for w and θ satisfying Eqns. (2.1)-(2.2), and (2.3)-(2.5) implies that the Rayleigh number R in terms of a, $\alpha_2 T_0$, K_0 and K_1 is given by

$$\begin{aligned}|R| = {} & \frac{1}{|1-\alpha_2 T_0|}\times\frac{10}{7\{K_0(K_1+9)+9(K_1+8)\}^2}\times \\ & \big[504\{K_0(K_1+4)+4(K_1+3)\}\{K_0(K_1+9)+9(K_1+8)\} \\ & +24a^2[72\{K_1(K_1+13)+51\}+3K_0\{5K_1(K_1+14)+312\}+K_0{}^2\{K_1(K_1+15)+72\}] \\ & +a^4[76K_1(K_1+15)+K_0\{17K_1(K_1+16)+1140\}+K_0{}^2\{K_1(K_1+17)+76\}+4464]\big].\end{aligned} \tag{3.14}$$

Remark. It is easily seen from equation (3.14) that the results analogous to those given in Remarks 1 to 3 are also valid, in the present case.

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